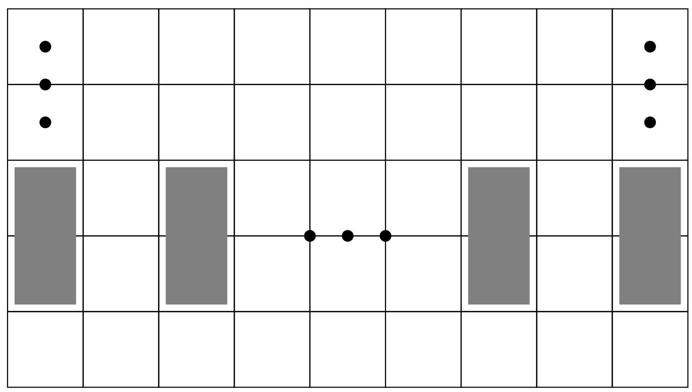
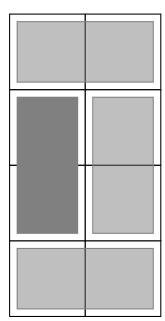
Solution

1. a) It's not always possible. The following counter example works:



The (0,0) square can only be tiled by a domino covering (0,0) and (0,1). Next, the (0,2) square can only be tiled by a domino covering (0,2) and (0,3). This continues by induction for squares (0,2k) covered by (0,2k) and (0,2k+1) dominos; however, when we get to the (0,n) square, it will be impossible to tile.

b) It is possible. Divide the board into 2x2 squares. If a square contains a whole domino, just tile the rest with another domino. If a domino is split between two 2x2 squares, then these two squares contain no other dominos. Now tile these pairs in the following manner:



2. We will use the following identity $F_n \cdot F_m + F_{n-1} \cdot F_{m-1} = F_{m+n-1}$ for the Fibonacci sequence. This identity is easily proven by induction on n. The basis for $n = 0$ and $n = 1$ are trivial, and the induction step is done by adding the identities for n and $n + 1$.

   We will use the following lemma: Let F_c be the smallest positive Fibonacci number divisible by p^d. Then $p^d \mid F_n$ if and only if $c \mid n$.

   Proof: The identity for $m = c + 1$ gives $F_n \cdot F_{c+1} + F_{n-1} \cdot F_c = F_{n+c}$. If $p^d \mid F_n$ and F_c, then it will divide F_{n+c}. This by induction proves $p^d \mid F_{kc}$. Now assume there exists n not divisible by c and $p^d \mid F_n$. Let $n = kc + r$, where $0 < r < c$. The identity provides: $F_{kc} \cdot F_{r+1} + F_{kc-1} \cdot F_r = F_{kc+r} = F_n$. $p^d \mid F_{kc}$, F_n, and F_{kc-1} is coprime to F_{kc} and hence coprime to p, therefore $p^d \mid F_r$. But this contradicts the assumption that F_c is the smallest Fibonacci number divisible by p^d.

   $2024 = 2^3 \cdot 11 \cdot 23$. The first Fibonacci number divisible by 8 is $F_6 = 8$ and the first divisible by 11 is $F_{10} = 55$. Finding the first Fibonacci number divisible by 23 is more challenging. We will write out the Fibonacci sequence modulo 23 until we get a zero:
   $0, 1, 2, 3, 5, 8, 13, 21, 11, 9, 20, 6, 3, 9, 12, 21, 10, 8, 18, 3, 21, 1, 22, 0, \ldots.$
   Hence the first Fibonacci number divisible by 23 is F_{22}. Now we use the lemma to conclude that 2024 divides F_n if and only if 6, 10, and 22 divide n. The smallest such n is $\mathrm{lcm}(6, 10, 22) = 330$.

3. The inequality can be changed to an equivalent form:

$$x^3y^3(x^2 + y^2 - 2) \geq (x + y)(xy - 1)$$

$$x^5y^3 + x^3y^5 + x + y \geq 2x^3y^3 + x^2y + xy^2$$

   Now use weighted AM-GM:

$$\frac{5x^5y^3 + 5x^3y^5 + 2x + 2y}{14} \geq x^3y^3$$

$$\frac{2x^5y^3 + 4x + y}{7} \geq x^2y$$

   Doubling the first inequality and adding it to the second and it's analogue provides the desired inequality.

4. Let D, E, F be the points where incircles of triangles ABC, ABX, ACX touch BC and let the other internal common tangent be s. We claim that s passes through D. Let D' be the point of intersection of the common tangent with BC. Notice that $D'I_1$ and $D'I_2$ are bisectors of BC and s. Therefore the angle $I_1D'I_2 = 90°$. Let the circle with diameter I_1I_2 be k, which intersects BC at two points X and D'. Let's prove $EX = DF$.

$$EX = \frac{BX + AX - AB}{2}$$

$$DF = CD - CF = \frac{CA + CB - AB}{2} - \frac{CX + CA - AX}{2} = EX$$

Since $OE = OF$ and $\angle OEX = \angle OFD$ then $\triangle OEX \cong \triangle OFD$. Hence $OX = OD$, so D belongs to k. D and D' coincide so s passes through a point which doesn't depend on X.