ENG

# THE MATHEMATICAL GRAMMAR SCHOOL CUP

## - MATHEMATICS -

**29. June 2022.**

This test consists of 12 problems on two pages. The problems are divided into two parts: multiple choice questions and "fill-in" problems where a student should fill in their answers by hand on the answer sheet. The examination lasts 180 minutes. The use of calculators, computers, or other electronic devices is strictly prohibited.

## PART ONE

*Problems 1 to 8 are multiple choice problems. Out of the five choices offered for a problem, exactly one is the correct answer. On the answer sheet you should circle only the letter that corresponds to the answer you have chosen. Each correct answer is worth 5 points.*

**1**. Let $B$ be the set of positive integers less than $10^6$ in the decimal representation of which the digits 2 or 7 appear. If $|B|$ is the number of elements of the set $B$, then the sum of its digits is:

(A) 18 (B) 30 (C) 36 (D) 42 (E) 45.

**2**. Suppose that $A$ is the set of all integer numbers $n$ such that the number $2^n + 2^8 + 2^{11}$ is a perfect square. Then the number of elements of set A is equal to:

(A) 0 (B) 1 (C) 2 (D) 4 (E) $+\infty$.

**3**. Let $d$ be the last digit of the number $x$, where $x$ is the least common multiple of numbers $2^{2^2} + 1, 2^{2^3} + 1, \dots, 2^{2^{2022}} + 1$, then $d$ is equat to:

(A) 1 (B) 4 (C) 7 (D) 8 (E) 9.

**4**. The angles at the vertices $A$ and $B$ of triangle $ABC$ are equal to $60°$ and $48°$, respectively. The line $p$, which contains the center of the inscribed circle of that triangle and is parallel to $AC$, intersects the line $AB$ at the point $P$. If $Q$ is the point on the segment $BC$, $BC = 3BQ$, then the $\measuredangle BPQ$ is equal to:

(A) $20°$ (B) $24°$ (C) $30°$ (D) $32°$ (E) $48°$.

**5**. Let $\mathbb{Q}$ be the set of all rational real numbers and let $f : \mathbb{Q} \to \mathbb{Q}$ be a function for which the following conditions hold:

(a) $(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$, for all $x, y \in \mathbb{Q}$,
(b) $f(1) = -1$.

The total number of all integer divisors of the number $f(10)$ is:

(A) 12 (B) 18 (C) 24 (D) 30 (E) 36.

**6**. Let $ABCD$ be a convex and cyclic quadrilateral. Suppose that the bisectors of the interior angles at the vertices $C$ and $D$ of that quadrilateral meet each other at the segment $AB$. If $AD = 5$ and $BC = 2$, then $AB(AD - BC)$ is equal to:

(A) 18 (B) 21 (C) 24 (D) 27 (E) 30.

**7**. Let $C$ be the set of all positive integers $n$ for which there is a polynomial $p_n$ of degree $n$, with integer coefficients, $p_n(0) = 0$, such that the equation $p_n(x) - n = 0$ has exactly $n$ integer solutions. Then $\max C - \min C$ is equal to:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4.

**8**. Let $ABCD$ be a convex quadrilateral of area 32. If for the lengths of the segments $AB$, $BD$ and $DC$ hold $AB + BD + DC = 16$, then $\frac{AC}{BD}$ is equal to:

(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) $2\sqrt{3}$ (D) $2\sqrt{2}$ (E) $\frac{3}{2}$.

## PART TWO

*Problems 9 to 12 are "fill-in" problems. Points for a certain part of the problem will be awarded only if this and all the answers to the previous parts are correct.*

**9**. If $a, b, c$ are positive real numbers such that $a^2 + b^2 + c^2 = 3$ prove that:

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \geq \frac{3}{2}$$

**10**. Let $ABC$ be a triangle and $k$ it's circumcircle. Let $k_a$ be the circle that touches the sides $AB$ and $AC$ and $k$ internally at $A_1$. Let $B_1$ and $C_1$ be defined analogously. Prove that $AA_1$, $BB_1$ and $CC_1$have a common point.

**11**. A tilling of a board consists of covering a board with given shapes such that all tiles are covered and no shapes overlap. We're covering a square $n \times n$ board with T and skew tetrominos. (T-tetromino consists of four squares in the shape of letter T, skew tetromino consists of 4 squares in the shape of letter S or Z)

(1) Can you tile a $6 \times 6$ board?
(2) Can you tile a $10 \times 10$ board?
(3) Can you tile a $2022 \times 2022$ board?

**12**. For which non-negative integers $n$ is $9^n + 10^n + 11^n$ a square of a natural number?

**GOOD LUCK!!!**