ENG

# THE MATHEMATICAL GRAMMAR SCHOOL CUP
# –MATHEMATICS–

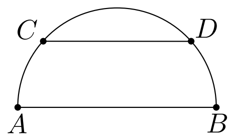
## Belgrade, June 23, 2020

The problems are divided into two parts: multiple choice questions and "fill-in" problems where a student should fill in their answers by hand on the answer sheet. The examination lasts 180 minutes.

## PART ONE

*Problems 1 to 8 are multiple choice problems. Out of the five choices offered for a problem, exactly one is the correct answer. On the answer sheet you should circle only the letter that corresponds to the answer you have chosen. Each correct answer is worth 5 points.*

**1.** The semicircle on the right has a radius of 10 cm, and chord $CD$ is parallel to diameter $AB$. If the length of $CD$ is $2/3$ of the length of $AB$, what is the distance between the chord and the diameter?




(A) $\frac{10\sqrt{5}}{3}$ cm; (B) $\frac{20\sqrt{5}}{3}$ cm; (C) $\frac{5\sqrt{5}}{3}$ cm; (D) $\frac{\sqrt{5}}{3}$ cm; (E) $\frac{2\sqrt{5}}{3}$ cm.

**2.** Nesha starts walking from point $A$ to point $B$. A half-hour later his friend Misha who walks 1 $km/h$ slower than twice Nesha's rate starts from the same point and follows the same path. If Misha overtakes Nesha after 2 hours of walking, how many kilometers will Nesha have covered at that time?

(A) $\frac{11}{5}$; (B) $\frac{10}{3}$; (C) 4; (D) 6; (E) $\frac{20}{3}$.

**3.** In an isosceles trapezium, the diagonals are perpendicular to each other. If the side length of the trapezium is equal to $2\sqrt{5}$ cm, and the ratio of the bases is $3:1$, then the area of the trapezium is equal to:

(A) 16 cm$^2$; (B) 15 cm$^2$; (C) $8\sqrt{3}$ cm$^2$; (D) $6\sqrt{5}$ cm$^2$; (E) 20 cm$^2$.

**4.** How many positive integers $n$ are there, so that $n^2 + 359$ is divisible by $n + 1$?

(A) 21 or less; (B) 22; (C) 23; (D) 24; (E) 25 or more.

**5.** Suppose that $a$ and $b$ are positive real numbers so that $a^2 - b^2 = 2ab = 4(a + b)$. Then $a$ equals to:

(A) $\sqrt{8} - 2$; (B) $6 + \sqrt{8}$; (C) 4; (D) $4 - \sqrt{8}$; (E) $4 + \sqrt{8}$.

**6.** A student is going to an amusement park. There are four attractions, pendulum ride, roller coaster, a carousel, and a water ride. The student buys 25 tokens. Each attraction costs 3 tokens per ride, except the roller coaster that costs 5. The student wants to try every attraction at least once, but the order of the rides does not matter. In how many ways can the student spend the tokens? There may be some remaining tokens at the end of the day; for example, the student can decide to take only four rides in total.

(A) 30; (B) 31; (C) 25; (D) 26; (E) 29.

**7.** How many pairs $(x, y)$ of solutions does the equation $x^{2020} = y^x$ have, if $x$ is a prime number and $y$ is a positive integer?

(A) 0; (B) 1; (C) 2; (D) 3; (E) 4 or more.

**8.** The integer $n$ consists of 2020 consecutive nines, $n = \underbrace{999\cdots999}_{2020}$. How many nines does the number $n^3$ contain in total?

(A) 2019; (B) 2021; (C) 2020; (D) 4040; (E) 4039.

## PART TWO

*Problems 9 to 12 are "fill-in" problems. Points for a certain part of the problem will be awarded only if this and all the answers to the previous parts are correct.*

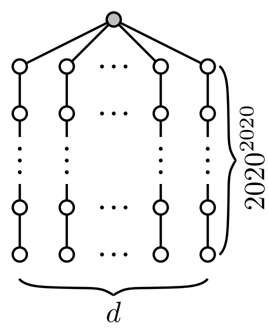
**9.** *[15 points]* Let $ABC$ be a triangle with $BC = 4$ cm, $CA = 9$ cm, $AB = 12$ cm, and let $I$ be the incenter of $\triangle ABC$. If $D$ is the other intersection of $CI$ with the circumcircle of $\triangle ABC$, and if $CD = 15,6$ cm, compute the length of $CI$.

**10.** Let $f(n)$ be a function that counts the number of positive integers bigger than $n$ and not larger than $n + 100$ that are divisible by 3 or 7, but not by both. Let $M$ be the maximal value of the function $f$ on the set $\{1, 2, 3, \ldots, 2020\}$ and let $k$ be the biggest positive integer not greater than 2020 such that $f(k) = M$.

(a) *[8 points]* Find $M$.
(b) *[7 points]* Find $k + f(k)$.

**11.** (a) *[2 points]* Find some numbers $b, c \in \mathbb{R}$ so that $|x^2 + bx + c| \leq \frac{1}{2}$ holds for all $x \in [-1, 1]$.
(b) *[3 points]* Find some numbers $b, c, e, f \in \mathbb{R}$ so that $|x^2 + bxy + y^2 + cx + ey + f| \leq 1$ holds for all $x, y \in [-1, 1]$.
(c) *[4 points]* Find some numbers $b, c, e, f \in \mathbb{R}$ so that $|x^2 + bxy + y^2 + cx + ey + f| \leq \frac{1}{2}$ holds for all $x, y \in [0, 1]$.
(d) *[6 points]* Find some numbers $a, b, c, d, e, f, g, h, i, j \in \mathbb{R}$ so that
$$|x^2y^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j| \leq \frac{1}{4}$$
holds for all $x, y \in [-1, 1]$.

**12.** Let $d$ be a positive integer. In a country called Matgimia there are $2020^{2020} \cdot d + 1$ towns that are connected by two-way roads in the following way: the capital of the country is linked to exactly $d$ other towns, while all other towns are organized in $d$ chains of length $2020^{2020}$ which start from the capital, as shown in the figure. Let $c(X)$ be the number of roads leaving town $X$ ($c(X) = d$ if $X$ is the capital, $c(X) = 1$ for the towns that are at ends of the chains, and $c(X) = 2$ otherwise). On January 1st, $n$ tourists arrived in the capital of Matgimia. On each of the following days, for each town $X$ in which there are at least $c(X)$ tourists, exactly $c(X)$ tourists leave that town and go to the neighboring towns, each one using a different road. After a while, it turned out that all the tourists stopped traveling and that there is at most one tourist in each of the towns. Let $n_1 < n_2 < n_3 < \cdots$ be all positive integers such that this situation is possible for $n = n_k$, $k = 1, 2, \ldots$. Find $n_{2020}$ if




(a) *[3 points]* $d = 1$; (b) *[3 points]* $d = 2$;
(c) *[4 points]* $d = 3$; (d) *[5 points]* $d = 2020$.