THE MATHEMATICAL GRAMMAR SCHOOL CUP
-MATHEMATICS-
BELGRADE, June 25, 2013

Part One

Problems 1 to 8 are multiple choice problems. Out of five offered choices for a problem, exactly one is the correct answer. The correct answer should be circled on the form.

1. Among the first thousand positive integers, how many are neither divisible by 4 nor by 6?

A) 625; B) 667; C) 584; D) 666; E) 416.

2. If $x+\frac{1}{x}=5$, then $x^2+\frac{1}{x^2}$ is equal to:

A) 15; B) 25; C) 23; D) 13; E) 20.

3. The equation $2|x+1|+x-3=0$ has:

A) infinitely many solutions; B) exactly three solutions; C) exactly two solutions;
D) exactly one solution; E) no solutions.

4. Medians AD and CE of triangle ABC intersect at point T. The midpoint of line segment AE is point F. The ratio of the areas of triangles TFE and ABC is equal to:

A) $1:12$; B) $1:8$; C) $1:9$; D) $1:6$; E) $1:16$.

5. The side length of square $ABCD$ is $a=1cm$. Let E and F, respectively, be the points on edges AD and AB such that $AE=AF$ and such that the area of rectangle $CDEF$ is maximal. Then the area of rectangle $CDEF$ is equal to:

A) $\frac{1}{2}cm^2$; B) $\frac{5}{8}cm^2$; C) $\frac{9}{16}cm^2$; D) $\frac{19}{32}cm^2$; E) $\frac{2}{3}cm^2$.

6. A ball with radius $13cm$ is intersected by a plane whose distance from the center of the ball is $5cm$. What is the area of this intersection?

A) $169\pi cm^2$; B) $25\pi cm^2$; C) $100\pi cm^2$; D) $144\pi cm^2$; E) $121\pi cm^2$.

7. How many four-digit numbers of the form $\overline{62**}$ are divisible by 90?

A) none; B) one; C) two; D) three; E) four.

8. How many integers satisfy the inequality $\dfrac{x+7}{\sqrt{9x^2+6x+1}}>2$?

A) infinitely many; B) none; C) exactly three; D) exactly two; E) exactly one.

Part Two

9. Find all positive integers m, n, and p such that $m+n+p=15$ and $(m-3)^3+(n-5)^3+(p-7)^3=540$.

10. Let x, y, and z be arbitrary positive real numbers. Show that the following inequality holds:

$$\frac{yz}{x^2+2yz}+\frac{zx}{y^2+2zx}+\frac{xy}{z^2+2xy}\le 1\le \frac{x^2}{x^2+2yz}+\frac{y^2}{y^2+2xz}+\frac{z^2}{z^2+2xy}.$$

11. The length of one edge of a tetrahedron is $4cm$, while the lengths of all other edges are equal to $3cm$. What is the volume of this tetrahedron?

12. Prove that there are infinitely many composite numbers among the numbers of the form 10^n+3 ($n=1, 2, 3, 4,\dots$).