THE MATHEMATICAL GRAMMAR SCHOOL CUP

-MATHEMATICS SOLUTIONS-

BELGRADE, June 26, 2019

PART ONE

The correct answers are: **1.** (A) **2.** (C) **3.** (B) **4.** (E) **5.** (B) **6.** (B) **7.** (D) **8.** (C)

PART TWO

9. Determine the last 3 digits of number $b = 1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot 26062019$.

Solution. First, note that $1000 = 2^3 \cdot 5^3 = 8 \cdot 125$. Number b is odd and it is clearly divisible by 125, hence the number $\overline{klm}$ formed by the last 3 digits of b is an element of the set $\{125,\ 375,\ 625, 875\}$. Recall that $b \equiv \overline{klm} \pmod 8$ and note that $125 \equiv 5 \pmod 8$, $375 \equiv 7 \pmod 8$, $625 \equiv 1 \pmod 8$, and $875 \equiv 3 \pmod 8$.
It is easy to see that, for $k \in \{0, 1, 2, 3, 4, 5, \ldots\}$,

$$(8k+1) \cdot (8k+3) \cdot (8k+5) \cdot (8k+7) \equiv 1 \cdot 3 \cdot 5 \cdot 7 \equiv 1 \pmod 8,$$

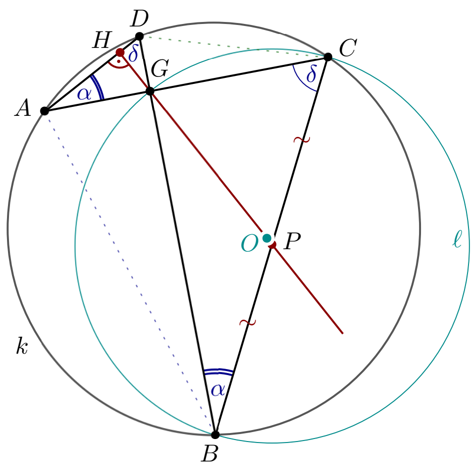
and, since $26062019 = 8 \cdot 3257752 + 3$, we get that

$$\begin{aligned} b &= (1 \cdot 3 \cdot 5 \cdot 7) \cdot (9 \cdot 11 \cdot 13 \cdot 15) \cdot \ldots \cdot 26062017 \cdot 26062019 \\ &\equiv 1^{3257752} \cdot 1 \cdot 3 \equiv 3 \pmod 8. \end{aligned}$$

Therefore, the last 3 digits of number b are 875. ▲

10. Let k be a circle and let AC and BD be two chords of different lengths which intersect in point G (A, B, C, D are distinct points). Let H be the foot of the perpendicular from point G to line segment AD. Line GH intersects line segment BC at point P so that $BP = PC$. Prove that $AC \perp BD$.

Solution.



Let $\sphericalangle DAC = \alpha$ and $\sphericalangle ADB = \delta$. Then we also have $\sphericalangle DBC = \alpha$ (inscribed angles subtended by chord DC) and $\sphericalangle ACB = \delta$ (inscribed angles subtended by chord AB).

Let ℓ be the circumscribed circle of triangle BCG and denote its center by O. Since H, G, and P are collinear, we have that

$$\sphericalangle BGP = \sphericalangle HGD = 90^\circ - \delta.$$

On the other hand, $\sphericalangle GOB = 2\sphericalangle GCB = 2\delta$ (inscribed and central angles in circle ℓ). Triangle $\triangle GOB$ is isosceles and therefore

$$\sphericalangle BGO = \frac{1}{2}(180^\circ - 2\delta) = 90^\circ - \delta.$$

We conclude that $\sphericalangle BGP = \sphericalangle BGO$, and points H, G, O, and P must be collinear.

Clearly $GB \neq GC$ (otherwise we would have $GD = GA$ as well, and then $AC = BD$, which cannot be since those chords are of different length). Now, point O lies on the bisector of segment BC, as well as on line GP. Since $GB \neq GC$, these two lines intersect in exactly one point, namely point P. Hence, $O \equiv P$ and

$$\sphericalangle BGC = \frac{1}{2}\sphericalangle BOC = \frac{1}{2} \cdot 180^\circ = 90^\circ.$$ ▲

11. Aleksa and Paja wrote 2019 positive integers on a blackboard. In one step, one can erase any two numbers a and b from the blackboard, and write (a, b) and $[a, b]$ instead (here, (a, b) denotes the greatest common divisor of a and b, and $[a, b]$ denotes their least common multiple). Prove that there exists a positive integer n such that, after n steps, the collection of numbers written on the blackboard cannot be changed anymore by using the procedure described above (the order in which the numbers are written on the blackboard is of no importance).

Solution. Denote by P the product of all the numbers Aleksa and Paja wrote on the blackboard. Since $a \cdot b = (a, b) \cdot [a, b]$, for any two positive integers a and b, substituting a and b with (a, b) and $[a, b]$ does not change the product of all numbers. Hence, each number that can be written on the blackboard at any point in time must be less than or equal to P, and the sum of all numbers cannot exceed $2019P$.

Let us show that, if $\{a, b\} \neq \{(a, b), [a, b]\}$, then $a + b < (a, b) + [a, b]$. Let $d = (a, b)$, and therefore $[a, b] = \frac{ab}{d}$. We have that $a, b > d$ and, therefore,

$$(a - d)(b - d) > 0 \iff ab - ad - bd + d^2 > 0$$
$$\iff ab + d^2 > ad + bd \iff \frac{ab}{d} + d > a + b.$$

It follows that, whenever we change the numbers written on the board, their sum strictly increases. We conclude that we cannot change the numbers infinitely many times (in fact, we cannot do it more than $2019P$ times). ▲

12. Find all triples (a, b, c) of real numbers so that:

$$\{a, b, c\} = \{ab + a + b, bc + b + c, ca + c + a\}.$$

Solution. If $\{a, b, c\} = \{ab + a + b, bc + b + c, ca + c + a\}$, then

$$\begin{aligned}\{a + 1, b + 1, c + 1\} &= \{ab + a + b + 1, bc + b + c + 1, ca + c + a + 1\}\\ &= \{(a + 1)(b + 1), (b + 1)(c + 1), (c + 1)(a + 1)\}.\end{aligned}$$

Let $x = a + 1$, $y = b + 1$, and $z = c + 1$. Our problem is equivalent to finding all real solutions to

$$A = \{x, y, z\} = \{xy, yz, zx\}.$$

If $|A| = 2$: Without loss of generality, assume that $x = y \neq z$. Then we have $\{x, z\} = \{x^2, xz\}$. Since $|A| = 2$, x cannot be equal to 0, so either $x = 1$ (from $x = x^2$) or $z = 1$ (from $x = xz$). The first case clearly works, and we get the set of solutions

$$(a, b, c) \in \{(0, 0, t), (0, t, 0), (t, 0, 0) \mid t \in \mathbb{R} \setminus \{0\}\}.$$

When $z = 1$, we get $\{x, 1\} = \{x^2, x\}$ and, since $x \neq z = 1$, $x^2 = 1$ and $x = -1$. Here we get

$$(a, b, c) \in \{(-2, -2, 0), (-2, 0, -2), (0, -2, -2)\}.$$

It is easy to check that all of these are indeed solutions of the stated problem.

If $|A| = 1$ or $|A| = 3$: We multiply and sum up all elements of set A:

$$(1) \qquad xyz = x^2y^2z^2,$$
$$(2) \qquad x + y + z = xy + yz + zx.$$

From the first equation we conclude that $xyz = 0$ or $xyz = 1$.

$xyz = 0$**:** At least one of x, y, z must be equal to zero, and hence, at least two of xy, yz, zx must be zero. We conclude that at least two of x, y, z must be equal to zero, and, following the same line of reasoning, $x = y = z = 0$.

The solution we obtain in this case is $a = b = c = -1$, which clearly satisfies the above condition.

$xyz = 1$**:** From equation (2) we get $xyz + x + y + z = xy + yz + zx + 1$. This is equivalent to $(x - 1)(y - 1)(z - 1) = 0$. Without any loss of generality, assume that $x = 1$. Now we have $\{1, y, z\} = \{1 \cdot y, yz, z \cdot 1\}$, i.e., $yz = 1$. It is easy to check that $(x, y, z) = (1, t, \frac{1}{t})$, for $t \in \mathbb{R} \setminus \{0\}$, satisfies that $\{x, y, z\} = \{xy, yz, zx\}$, and therefore in this case we get

$$(a, b, c) \in \left\{(0, t - 1, \frac{1}{t} - 1), (\frac{1}{t} - 1, 0, t - 1), (t - 1, \frac{1}{t} - 1, 0) \mid t \in \mathbb{R} \setminus \{0\}\right\}.$$

Finaly, the set of all solutions is

$$\begin{aligned}\mathcal{S} = \{(-1, -1, -1), &(0, 0, 0)\} \cup \{(0, 0, t), (0, t, 0), (t, 0, 0) \mid t \in \mathbb{R} \setminus \{0\}\}\\ &\cup \left\{(0, t - 1, \frac{1}{t} - 1), (\frac{1}{t} - 1, 0, t - 1), (t - 1, \frac{1}{t} - 1, 0) \mid t \in \mathbb{R} \setminus \{0, 1\}\right\}.\end{aligned}$$

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