THE MATHEMATICAL GRAMMAR SCHOOL CUP

-MATHEMATICS-

BELGRADE, June 23, 2015

PART ONE

The correct answers are: **1.** (B) **2.** (C) **3.** (E) **4.** (D) **5.** (A) **6.** (D) **7.** (C) **8.** (C)

PART TWO

1. Find all positive integers n such that the last digit (in the decimal representation) of the sum $1+2+3+\cdots+n$ is equal to 7.

Solution. Recall that $1+2+\cdots+n=\dfrac{n(n+1)}{2}$. If the last digit of $\dfrac{n(n+1)}{2}$ is 7, then the last digit of $n(n+1)$ has to be equal to 4. Now we have the following possibilities:

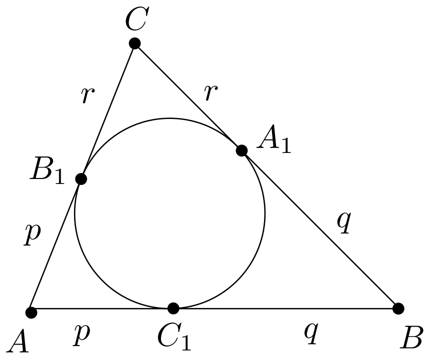
the last digit of n	0	1	2	3	4	5	6	7	8	9
the last digit of $n+1$	1	2	3	4	5	6	7	8	9	0
the last digit of $n(n+1)$	0	2	6	2	0	0	2	6	2	0

We see from the table that the last digit of the product of two consecutive positive integers cannot be equal to 4, which means that there is no solution to this problem.

2. Let A_1, B_1, and C_1 be the touching points of the inscribed circle of a triangle ABC with its sides $BC=a$, $AC=b$, and $AB=c$, respectively, and let $AC_1=p$, $BA_1=q$, and $CB_1=r$. Prove that the following inequality holds:

$$\frac{p}{a}+\frac{q}{b}+\frac{r}{c}\geq\frac{3}{2}.$$

When does the equality hold?



Solution. It is well known that

$$p=\frac{c+b-a}{2},\ q=\frac{c+a-b}{2},\ r=\frac{a+b-c}{2}.$$

If we rewrite our inequality, we see that we ought to prove that

$$\frac{c+b-a}{2a}+\frac{c+a-b}{2b}+\frac{a+b-c}{2c}\geq\frac{3}{2}.$$

The left hand side of the inequality is equal to

$$L=\frac{1}{2}\left(\frac{c}{a}+\frac{b}{a}-1\right)+\frac{1}{2}\left(\frac{c}{b}+\frac{a}{b}-1\right)+\frac{1}{2}\left(\frac{a}{c}+\frac{b}{c}-1\right)=\frac{1}{2}\left(\frac{c}{a}+\frac{a}{c}+\frac{b}{a}+\frac{a}{b}+\frac{c}{b}+\frac{b}{c}-3\right).$$

Using the fact that, for $t>0$, $t+\dfrac{1}{t}\geq 2$ (here the equality holds if and only if $t=1$) we get

$$L\geq\frac{1}{2}(2+2+2-3)=\frac{3}{2}.$$

It is now easy to see that $\dfrac{c+b-a}{2a}+\dfrac{c+a-b}{2b}+\dfrac{a+b-c}{2c}=\dfrac{3}{2}$ if and only if $\dfrac{a}{c}=\dfrac{b}{c}=\dfrac{a}{b}=1$, i.e., if and only if ABC is an equilateral triangle.

3. Show that there is a number of the form $\underbrace{77\ldots77}_{n-\text{sevens}}$ which is divisible by 2017.

Solution 1. We will prove that an even stronger statement is true: one of the numbers that can be written using at most 2017 sevens is divisible by 2017. Assume the contrary is true, i.e., that none of the numbers from the set

$$\{7, 77, 777, \ldots, \underbrace{77\ldots77}_{2017}\}$$

is divisible by 2017. Since we have 2016 possible non-zero remainders when a number is divided by 2017, by using the Pigeon-Hole Principle, we conclude that at least two of these numbers have the same remainder. Let those be

$$n_i = \underbrace{77\ldots77}_{i \text{ sevens}} \text{ and } n_j = \underbrace{77\ldots77}_{j \text{ sevens}}, \text{ where } i < j.$$

We now have that $2017 \mid n_j - n_i$. In other words, for some integer k,

$$2017k = n_j - n_i = \underbrace{77\ldots7}_{j-i}\underbrace{0\ldots00}_{i} = 10^i \cdot \underbrace{77\ldots7}_{j-i}.$$

Since $\gcd(2017, 10) = 1$, we conclude that $2017 \mid \underbrace{77\ldots7}_{j-i}$.

Solution 2. Since 2017 is a prime number, by the Little Fermat's Theorem, $10^{2016} \equiv_{2017} 1$. Therefore,

$$9 \cdot \underbrace{77\ldots7}_{2016} = 9 \cdot 7 \cdot \left(10^{2015} + 10^{2014} + \cdots + 10 + 1\right) = 7 \cdot \left(10^{2016} - 1\right) = 2017k, \; k \in \mathbb{N}.$$

Since $\gcd(2017, 9) = 1$, we get that $2017 \mid \underbrace{77\ldots7}_{2016}$.

4. Mariana decided to study for 23 days in June and to relax during the rest of the days. In how many ways can Mariana choose the seven days to relax if she decided never to relax for two days in a row?

Solution. Let $1 \le d_1 < d_2 < \cdots < d_7 \le 30$ be one choice of days satisfying the conditions. Note that the requirement that there are no two consecutive numbers among the d_j's and $1 \le d_j \le 30$ for all j is equivalent to the requirement that

$$1 \le d_1 < d_2 - 1 < d_3 - 2 < d_4 - 3 < d_5 - 4 < d_6 - 5 < d_7 - 6 \le 30 - 6 = 24.$$

In other words, choosing days $1 \le d_1 < d_2 < \cdots < d_7 \le 30$ satisfying the conditions of our problem is the same as choosing any seven numbers $d'_1 = d_1,\ d'_2 = d_2 - 1, \ldots, d'_k = d_k - k + 1, \ldots, d'_7 = d_7 - 6$ from the set $\{1, 2, 3, \ldots, 23, 24\}$. This can be done in

$$\frac{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18}{7!} = \binom{24}{7} \text{ different ways.}$$

Note. We divide by 7! because there is a unique way to arrange seven different chosen numbers in an increasing sequence $d'_1, d'_2, \ldots, d'_7$.